

# Integrability in non-perturbative QFT<sup>1</sup>

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## ABSTRACT

Exact non-perturbative partition functions of coupling constants and external fields exhibit huge hidden symmetry, reflecting the possibility to change integration variables in the functional integral. In many cases this implies also some non-linear relations between correlation functions, typical for the tau-functions of integrable systems. To a variety of old examples, from matrix models to Seiberg-Witten theory and AdS/CFT correspondence, now adds the Chern-Simons theory of knot invariants. Some knot polynomials are already shown to combine into tau-functions, the search for entire set of relations is still in progress. It is already known, that generic knot polynomials fit into the set of Hurwitz partition functions – and this provides one more stimulus for studying this increasingly important class of deformations of the ordinary KP/Toda  $\tau$ -functions.

## 1 Introduction

For a long time integrability was thought to be an exotic phenomenon, useful for providing solvable examples in the textbooks and for enjoying the intellectual gourmards by mathematical beauties and possibilities to tie together the different branches of science.

However, today, despite the growing importance of these motivations, especially the last one – actually, the key goal of the *string theory* [1], – integrability occupies a much bigger place in theoretical studies, and its role and significance is only increasing. The word “integrability” appeared hundreds of times only in the titles of the hep-th papers during last several years, well known subjects include matrix models, Seiberg-Witten theory of non-perturbative Yang-Mills fields, AdS/CFT-correspondence and many newly-emerging topics – Hurwitz and knot theories among the most recent ones. Ironically, when one performs this search in arXiv, it lists together the titles with “integrability” and “integrals”, predominantly functional. There it is, probably, just the linguistics, but in fact this is symbolic, because integration is indeed the place, where the modern role of integrability has its origin.

In fact, today we think [1] that integrability is the pertinent feature of non-perturbative (i.e. exact) partition functions – the quantities obtained by evaluation of functional integrals, and depending on the parameters of the integration measure (coupling constants), boundary conditions (background fields and vacuum condensates) and the integration domain (cut-offs). The main property of the integral is that it does not depend on the integration variables – and this is the property, which is finally reflected in integrability features of the partition function, often not immediately visible (hidden), when it is calculated by various techniques, which (most of them) do not explicitly respect integrability.

## 2 Renormalization group

Traditional way to think about integrable systems is in terms of commuting Hamiltonian flows on the phase space – but this is not at all straightforward to see and identify these flows in the study of non-perturbative physics. Perhaps, the most obvious place where at least *one* flow is studied for many years in this context, is the renormalization group.

Namely, when one tries to define a partition function like

$$Z(g_k|\Lambda) = \int_{|p|<\Lambda} D\phi(x) e^{iS\{\phi\}/\hbar} \quad (1)$$

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where, morally,

$$S\{\phi\} = \int \left( (\partial\phi)^2 + \sum_k g_k \phi^k \right) d^d x \quad (2)$$

it is possible to ask that the coupling constants  $g_k$  depend on the cut-off  $\Lambda$  in such a way, that

$$\frac{d}{d\Lambda} Z(g_k(\Lambda)|\Lambda) = 0 \quad (3)$$

This *renormalization procedure* is extremely interesting by itself, and itself involves a lot of group theory and integrable-like structures [2, 3] – but this is beyond the scope of this text. What matters, this principle defines a flow

$$\frac{\partial g_k}{\partial \Lambda} = \beta_k(g) \quad (4)$$

on the space of the coupling constants. Of course, it is just a single flow, to make it into a set of flows one should consider arbitrary deformations of the boundary (cut-off) of the integration domain [4, 5], what was never really studied in quantum field theory.

Still, even looking at one flow, one can say something. The point is that the space of coupling constants is multi-dimensional, and in such case a generic flow is "chaotic". This is what one *could* expect from generic renormalization group [6], but this is what was yet never observed in any QFT example. Partly this is reflected in various versions of Zamolodchikov's *c*-theorem [7], but actually the statement is stronger: the flow could be contractive, but still chaotic in orthogonal directions. The fact that this does not seem to happen, can be a signal of a hidden integrability – and in a context, much broader than any of the already well-studied examples.

### 3 Ward identities and AMM/EO topological recursion

Still, the flows are not the most convenient language to describe integrability in modern days. It usually shows up differently: as an infinite set of consistent differential equations on partition function  $Z(g_k)$ . These equations are in fact nothing but Picard-Fuchs equations for the integrals, which depend on parameters – and they reflect the possibility to change integration variables, what can be alternatively imitated as the change of the coupling constants. In QFT such equations are also known as Ward identities.

The archetypical example [8] is the matrix integral (matrix model) – with already many enough integration variables to observe non-trivial structures, and with still few enough to make as detailed calculations as one can wish. This calculation leads to the celebrated *Virasoro constraints a la* [9]:

$$\hat{L}_n Z(t) = \sum_k k(T_k + t_k) \frac{\partial Z}{\partial t_{k+n}} + \hbar^2 \sum_{a+b=n} \frac{\partial^2 Z}{\partial t_a \partial t_b} = 0, \quad n \geq -1 \quad (5)$$

which express invariance of

$$Z(t_k) = \int_{N \times N} dM e^{\frac{1}{\hbar} (-\text{Tr } W(M) + \sum_k t_k \text{Tr } M^k)} \quad (6)$$

under the change  $\delta M = \epsilon M^{n+1}$ . This allows to define the partition function as a *D*-module, for which (6) is just an integral representation. As to the integral, it can be well defined by introduction of background potential  $W(z) = -\sum_k T_k z^k$ , usually polynomial, for example, quadratic.

Amusingly, the dependence on this, originally auxiliary, potential appears to be one of the most interesting [10]. The terms of the *genus expansion* of the logarithm  $\log Z$  of the partition function in powers of t'Hooft coupling constant  $\hbar^2 N$ , are expressed as the meromorphic poly-differentials on a hyperelliptic Riemann surface,

$$y^2 = W'(z)^2 - f(z) \quad (7)$$

defined by the polynomial  $W(z)$  and some additional data  $f(z)$  – hidden in the naive definition (6). Virasoro constraints then turn into a recurrent set of relations, actually building a set of poly-differentials on the families of Riemann surfaces (often called *spectral curves*), which is now known under the name of AMM/EO topological recursion [10, 11] and which is now being found in a variety of different fields, from Seiberg-Witten theory of instanton sums [12, 13, 14] to knot theory [15]. From integrability-theory point of view this phenomenon – emergence of a new recursions from expansions of a *D*-module – is an example of emerging Whitham hierarchies, to be briefly described in s.6 below.

In *M*-theory context the spectral curve can be nicely visualized as a surface, on which the *6d* theory is compactified [23], and this picture leads to various deep insights [24], including the AGT conjecture [25].

## 4 Group theory and quadratic relations

Ward identities are *linear* relations, imposed on exact non-perturbative correlators. Topological recursion looks quadratic, but this is only a result of rewriting the action of quadratic operator on the exponential – the underlying equations are still essentially linear (though this is not so easy to see in some recent applications of the topological recursion).

Remarkably, non-perturbative partition functions can also satisfy *truly* quadratic relations, which have nothing to do with linearity. They are usually called Hirota-like equations, and functions, that satisfy such equations, are known as  $\tau$ -functions. The best known is KP  $\tau$ -function, satisfying

$$\oint dz e^{\sum_k (t_k - t'_k) z^{-k}} \tau \left( t_k + \frac{z^k}{k} \right) \tau \left( t'_k - \frac{z^k}{k} \right) = 0 \quad (8)$$

what, if expanded in powers of  $t_k - t'_k$ , provides an infinite set of quadratic differential equations (KP hierarchy):

$$3(\tau_{22}\tau - \tau_2^2) - 4(\tau_{13}\tau - \tau_1\tau_3) + (\tau_{1111}\tau - 4\tau_{111}\tau_1 + 3\tau_{11}^2) = 0, \quad \dots \quad (9)$$

The origin of such equations [16] is in the theory of Lie groups, where one can multiply representations

$$R_1 \otimes R_2 = \oplus_i R_i \quad (10)$$

i.e. there is a comultiplication and intertwining operators. Quadratic relations appear when one consider two different decompositions with some common representation  $Q$ , belonging to both:

$$R_1 \otimes R_2 \longrightarrow Q \longleftarrow R_3 \otimes R_4 \quad (11)$$

Then a system of quadratic relations can be written on matrix elements of the group elements [17], in different representations – and this will be a Hirota-like equation for a generating function of such matrix elements.

Unfortunately, not too many examples are worked out so far, beyond the KP/Toda family, which is associated to totally antisymmetric representations of the  $\widehat{U(1)}$  algebra. Remarkably, a new interest is now emerging to a quasiclassical limit of this construction for quantum groups – it is related to cluster algebras and to the Seiberg-Witten theory [18].

A very interesting fact about the tau-functions is that they can often be described in terms of auxiliary Riemann surfaces – spectral curves. In other words, the Liouville tori of integrable systems are often Jacobians of Riemann surfaces, and complex moduli are invariants of the Hamiltonian flows. It is still not clear, how general this phenomenon is, but it definitely holds for KP/Toda families [19].

## 5 Interplay between the linear and quadratic relations

Thus non-perturbative partition functions naturally satisfy rich sets of *linear* Ward identities – as a result of the freedom to choose integration variables in the functional integral – and *quadratic* relations – as a result of existence of operator formalism in quantum mechanics, where amplitudes are matrix elements. With this understanding of their origins, it is clear, that the connection between linear and quadratic relations is not going to be very straightforward – as is the case of the subtle connection between the two formalisms in quantum theory. Indeed, even relation between the Virasoro constraints (5) and KP Hirota equation (8) for the simple matrix model (6) is not fully understood.

In any case, it is clear that non-perturbative partition functions should be some kind of the tau-functions, but definitely not of generic type: in addition to *quadratic* they should satisfy at least one *linear* constraint – then all other Ward identities will follow. This single constraint is often called *string equation*, and such solutions to Hirota-like equations are called *matrix-model  $\tau$ -functions* [20].

## 6 Whitham hierarchies and Seiberg-Witten theory

Normal in quantum field theory is not to define partition functions by integrating over all the fields completely, but rather do so with short-distance fluctuations, so that the answer remains dependent on the background slowly-varying fields, and effective action contains only terms with few derivatives. Such procedure gets especially important, because we believe that the Standard Model of the fundamental interactions arises in exactly

this way from some still unknown UV-complete theory at Planckian scale (the superstring model [21] being one of the most popularized candidate, provided in the string theory context [1]). This procedure has, of course, a counterpart in integrability theory, where averaging over fast variables converts ordinary integrable hierarchies, satisfying Hirota equations, into effective theories of slow modes, described by the *Whitham theory* [22]. The crucial feature of Whitham hierarchy is that it depends on the choice of solution to original (high energy) hierarchy, and in the case when those solutions are parameterized by spectral curves, – it depends on the spectral curve.

In other words, the belief is that the low-energy effective actions should be somehow described in terms of some additional hidden structure – auxiliary Riemann surfaces, or, better, the families of such surfaces, parameterized by the choice of the vacuum. Whitham dynamics defines the dependence of the low-energy effective action on the moduli – and the result is that it is often constrained by a new kind of non-linear equations: the *WDVV-like system* [26]

$$\mathcal{F}_i \mathcal{F}_j^{-1} \mathcal{F}_k = \mathcal{F}_k \mathcal{F}_j^{-1} \mathcal{F}_i \quad (12)$$

where  $(\mathcal{F}_i)_{jk} = \partial_{ijk}^3 \log \mathcal{Z}$  are matrices, made out of the third derivatives of the low-energy *prepotential*  $\mathcal{F} = \log \mathcal{Z}$ , and derivatives are with respect to some special coordinates  $\{a_i\}$  on the moduli space of spectral curves, introduced via the Seiberg-Witten (or special-geometry) equations

$$\oint_{A_i} \Omega = a_i, \quad \oint_{B_i} \Omega = \partial_i \mathcal{F} = \frac{\partial \log \mathcal{Z}}{\partial a_i} \quad (13)$$

This construction appeared to be extremely successful [27] in application to the only known explicit example of exact non-perturbative effective action: the one obtained by summation over instantons in the  $N = 2$  supersymmetric Yang-Mills theory in 3,4,5 and 6 space time dimensions [13, 14]. In fact, it establishes a one-to-one correspondence between various quiver gauge theories and 1-dimensional integrable systems, like Lioville/Toda systems and spin chains, provides an effective description of  $S$ -dualities [24, 28] and relates them to non-trivial spectral dualities between integrable systems, like [29, 30].

## 7 Lifting Whitham prepotentials to $\tau$ -functions

Of course, low-energy effective action is only a small remnant of original non-perturbative partition function. Likewise the Whitham prepotential is only a small remnant of original  $\tau$ -function, and there can be many different non-perturbative partition functions, leading to the same prepotential. It can be natural to look for the *minimal* among such extensions.

In the case of the Seiberg-Witten theory, where Whitham dynamics is associated with entire world of 1-dimensional integrable systems, such an extension should be also distinguished. Today we already know that the corresponding minimal partition functions are closely related with conformal blocks of  $2d$  conformal field theories – this is the celebrated AGT relation [25, 31]. The best possible formulation of it is in terms of Hubbard-Stratanovich duality in a peculiar Dotsenko-Fateev matrix model [32], where Seiberg-Witten spectral curve and associated genus expansions are naturally appearing. A particular subset (particular Nekrasov-Shatashvili limit) of these conformal blocks is related to quantization of the  $1d$  integrable systems [33].  $S$ -duality relations seem to be realized in terms of  $3d$  Chern-Simons theory.

## 8 Chern-Simons theory and knot polynomials

After distinguished  $1d$  and  $2d$  quantum field theories found their place in the world of universality classes of non-perturbative partition  $\tau$ -functions, the  $3d$  Chern-Simons theory [34] is the natural next model to consider. And indeed the growing efforts are now applied in this direction.

Chern-Simons theory is a version of Yang-Mills theory, with a topological action:

$$\int DA \, e^{\frac{1}{\hbar} \int_{\mathcal{M}_3} (AdA + \frac{2}{3} A^3)} \quad (14)$$

and the observables are gauge invariant Wilson-loops

$$\text{Tr}_R \, P \exp \oint_{\mathcal{K}} A \quad (15)$$

along various lines (knots)  $\mathcal{K}$  embedded into a three-dimensional manifold  $\mathcal{M}_3$ . The Wilson-loop averages are often called *knot functions*, they depend on the space  $\mathcal{M}_3$ , on the knot (or the link)  $\mathcal{K}$ , on the gauge group  $G$ , on its representation  $R$  and on the coupling constant  $\hbar$ . For the simply connected  $\mathcal{M}_3 = R^3$  or  $S^3$  and for  $G = Sl(N)$  the knot functions are actually *polynomials* of the variables  $q = e^{2\pi i \hbar}$  and  $A = q^N$  – and they are often called *HOMFLY polynomials*:

$$H_{\otimes_i R_i}^{\cup_i \mathcal{K}_i \in \mathcal{M}_3}(q|A) = \left\langle \prod_i \text{Tr}_{R_i} P \exp \oint_{\mathcal{K}_i} A \right\rangle_{CS} \quad (16)$$

There are different ways to carefully define and evaluate HOMFLY polynomials, leading to various kinds of important quantities and structures, see [35] for classical results and [36] for relatively fresh reviews. In particular, if knots/links are represented by the closure of braids in the projection to the two-dimensional plane (this happens if the functional integral is evaluated in the temporal gauge  $A_0 = 0$ ), then knot polynomials exhibit a character decomposition [37, 38, 39, 40]:

$$H_R^\mathcal{K}(q|A) = \sum_Q C_{RQ}^\mathcal{K}(q) \chi_Q(q|A) \quad (17)$$

which completely separates knot ( $\mathcal{K}$ ) and group ( $A$ ) dependencies. In the case of HOMFLY polynomials the expansion basis  $\chi_Q$  is provided by Schur functions (the characters of  $Sl(\infty)$ ), and this implies a natural deformation to MacDonald polynomials – such deformation of knot polynomials, depending on two parameters  $q$  and  $q'$ , indeed exists and leads [39, 42] to *superpolynomials* [41, 43]. Character expansion is actually defined on a bigger space of time variables instead of just  $A$ , where one introduces *extended knot polynomials* [39, 44],

$$\mathcal{P}_R^\mathcal{K}(q, q'|t_k) = \sum_Q C_{RQ}^\mathcal{K}(q, q') \chi_Q(q, q'|t_k) \quad (18)$$

which can be directly compared to  $\tau$ -functions. It turns out that – after appropriate summation over representations  $R$  – they indeed give rise to  $\tau$ -functions in the case of the *torus knots* [44], but for generic knots this is not true: some modification is needed, either of the extension of knot polynomials, or of the  $\tau$ -functions, perhaps, taking into account the existence of "quantum" parameter  $q$ .

From the point of view of integrability, the most interesting seems the *genus expansion* near the point  $q = 1$ . Exactly at this point the HOMFLY polynomial exhibits a very simple dependence on the representation:

$$H_R^\mathcal{K}(q = 1|A) = \left( \sigma_\square^\mathcal{K}(A) \right)^{|R|} \quad (19)$$

where  $|R|$  is the number of boxes in the Young-diagram  $R$ , and  $\square$  denotes the fundamental representation, represented by a diagram with a single box. However, genus expansion is far more interesting [45]: the first correction is

$$H_R^\mathcal{K}(q|A) = \left( \sigma_\square^\mathcal{K}(A) \right)^{|R|} + (q - q^{-1}) \varphi_R([2]) \left( \sigma_\square^\mathcal{K}(A) \right)^{|R|-2} \sigma_{[2]}^\mathcal{K}(A) + \dots \quad (20)$$

and in general

$$H_R^\mathcal{K}(q|A) = \left( \sigma_\square^\mathcal{K}(A) \right)^{|R|} \exp \left( \sum_j (q - q^{-1})^j \sum_{|Q| \leq j+1} \varphi_R(Q) \frac{\sigma_{j|Q}^\mathcal{K}(A)}{(\sigma_\square^\mathcal{K}(A))^{2j}} \right) \quad (21)$$

Here  $\sigma_{j|Q}(A)$  are various *special polynomials* in  $A$ , with coefficients made out of *Vassiliev invariants* [46] of the knot  $\mathcal{K}$ . The  $R$ -dependent coefficients  $\varphi_R(Q)$  are the characters of symmetric group  $S_\infty$  – which are nowadays studied in the framework of the *Hurwitz theory*. Note that for a given order  $j$  of the genus expansion only Young diagrams of the sizes  $|\Delta| \leq j + 1$  contribute, if one did not exponentiate the expansion, analogous restriction would be weaker:  $|\Delta| \leq 2j$ .

Extremely interesting seems the generalization of genus expansion (21) to superpolynomials, of which only the first-order term is conjectured [47, 48]:

$$P_R^\mathcal{K}(q, q'|A) = \left( P_\square^\mathcal{K}(q, q'|A) \right)^{|R|} + \left( (q - q^{-1}) \nu_{R'} - (q' - q'^{-1}) \nu_R \right) \sigma_{[2]}^\mathcal{K}(A) \left( \sigma_\square^\mathcal{K}(A) \right)^{|R|-2} + \dots \quad (22)$$

where  $\varphi_R([2]) = \nu_{R'} - \nu_R$ ,  $R'$  is transposed diagram  $R$ , and  $\nu_R = \sum_j (j-1)r_j$ . It should also include a non-trivial deformation of Hurwitz theory, describing *refinement* of symmetric group characters  $\varphi_R(Q)$ , which still remains to be found.

An intriguing question is the consistency between (22) and the desired positivity of the colored superpolynomial: beyond non-(anti)symmetric representations the relation can be more involved [48], while (22) can instead describe MacDonald-inspired not-always-positive colored superpolynomials of [39, 42].

## 9 Hurwitz partition functions

The central object in Hurwitz theory [49] is the set of commuting operators  $\hat{W}_R$ , parameterized by Young diagrams  $R = \{r_1 \geq r_2 \geq \dots \geq 0\}$  – labeling the conjugation classes of permutations of  $|R|$  elements (see [50] for a non-commutative extension, associated with open rather than closed string theory). These operators have  $Sl(\infty)$  characters (Schur functions) as their common eigenfunctions, while the corresponding eigenvalues are the symmetric group characters  $\varphi_R(Q)$ :

$$\hat{W}_Q \chi_R = \varphi_R(Q) \chi_R \quad (23)$$

The simplest is representation of the operators in Miwa coordinates  $t_k = \frac{1}{k} \text{Tr } X^k$ , then

$$\hat{W}_R \sim : \prod_i \text{Tr} \left( X \frac{\partial}{\partial \bar{X}} \right)^{r_i} : \quad (24)$$

In ordinary time-variables the first non-trivial is the *cut-and-join operator*

$$\hat{W}_{[2]} = \sum_{a,b} \left( ab t_a t_b \frac{\partial}{\partial t_{a+b}} + (a+b) t_{a+b} \frac{\partial^2}{\partial t_a \partial t_b} \right) \quad (25)$$

which appears in various branches of science, including the construction of Khovanov knot polynomials [51].

Relation to Hurwitz numbers (counting ramified  $n$ -sheet genus- $g$  coverings of Riemann surfaces) is through the Frobenius formula [52]

$$\text{Cover}_{n,g}(Q_1, \dots, Q_p) = \sum_{|R|=n} d_R^{2-2g} \varphi_R(Q_1) \dots \varphi_R(Q_p) \quad (26)$$

It implies consideration of the generating functions of the form (at genus  $g = 0$ )

$$Z(\beta|t) = \exp \left( \sum_Q \beta_Q \hat{W}_Q \right) e^{t_1} = \sum_R \exp \left( \sum_Q \beta_Q \varphi_R(Q) \right) \chi_R(t) \quad (27)$$

which we call *Hurwitz partition functions*.

It is clear from (21) that HOMFLY polynomials are actually the Hurwitz partition functions, evaluated at some knot-dependent point in the space of  $\beta$ -variables. They can actually be rewritten in terms of the operators  $\hat{W}$ , if one considers the Ooguri-Vafa generating function

$$Z_{OV}^{\mathcal{K}}(t) = \sum_R H_R^{\mathcal{K}} \chi_R(t) \quad (28)$$

– note that this introduces a complementary set of times to (18). Then (21) implies that

$$Z_{OV}^{\mathcal{K}}(t) = \exp \left( \sum_j (q - q^{-1})^j \sum_{|Q| \leq j+1} \frac{\sigma_{j|Q}^{\mathcal{K}}(A)}{(\sigma_{\square}^{\mathcal{K}}(A))^{2j}} \hat{W}_Q \right) \exp \left( \sum_k \frac{(A^k - A^{-k}) \cdot (\sigma_{\square}^{\mathcal{K}}(A))^k}{q^k - q^{-k}} t_k \right) \quad (29)$$

It was conjectured [53] that the character expansion of  $\log Z_{OV}$  possesses non-trivial *integrability* properties (not to be mixed with *integrability*, at least immediately), which can be understood with the help of the theory of  $\hat{W}$  operators [54].

Since operators  $\hat{W}_Q$  form a *commutative* algebra,  $Z(\beta|t)$  – and thus  $Z_{OV}^{\mathcal{K}}(t)$ , as its particular case – is clearly related to *integrable* systems. However, only for the unknot  $Z_{OV}$  turns out to be an ordinary KP  $\tau$ -function, satisfying Hirota equation (8) and possessing a free-fermion representation. For a general argument *against* such naive integrability for arbitrary *knots* (but not *links*) see footnote 3 in [44]. In fact, integrability properties of Hurwitz partition function are not fully understood and their connection to KP/Toda integrability remains obscure. In particular, KP integrability in  $t$ -variables is preserved only by the action of Casimir operators, i.e. when the  $\hat{W}_Q$  operators in (27) are *linearly* combined to form integrability-preserving *Casimir operators*, what does not happen for arbitrary values of  $\beta_Q$  [55]. This can mean that either some change of  $t$ -variables is required to make integrability visible, or that KP-integrability is somehow deformed or generalized when one moves from  $2d$  conformal theories to  $3d$  Chern-Simons.

## 10 Conclusion

Integrability proved to be an extremely important part of quantum field theory: its main role is to describe full non-perturbative answers in quantum field theory – and this fact is deeply related to the very essence of quantum theory, whether it is formulated in operator or in functional integral formalism. It, however, remains an open question, what is exactly the right version of integrability theory which matters. Absolute majority of the known examples involves the simplest KP/Toda integrability, associated with the free fermion (antisymmetric) representations of the group  $\widehat{U}(1)$ , though traces of more general quantum groups are already seen. Today the frontline seems to be at the study of Hurwitz partition functions, of which knot polynomials provide a vast set of examples: it is crucially important to understand and effectively describe their properties. Despite not immediately belonging to the set of KP/Toda  $\tau$ -functions, they seem to be at least their closest relatives, moreover, they seem to possess genus expansions, described in terms of the spectral curves and AMM/EO topological recursion.

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## References

- [1] A.Morozov, *Rus.Physics Uspekhi* 35 (1992) 67
- [2] A.Connes and D.Kreimer, *CMP* 210 (2000) 249-273, hep-th/9912092; hep-th/0003188; hep-th/0510202
- [3] A.Gerasimov, A.Morozov and K.Selivanov, *Int.J.Mod.Phys. A10* (1995) 2589-2614, hep-th/0005053; D.Malysh, *JHEP* 0205 (2002) 013, hep-th/0112146; *TMP* 143 (2005) 505-514, hep-th/0408230; A.Morozov and M.Serbyn, *Theor.Math.Phys.* 154 (2008) 270-293, hep-th/0703258
- [4] J.Polchinsky, *Nucl. Phys. B231* (1984) 269; *String Theory*, Vol.1-2, Cambridge University Press, 1999.
- [5] A.Mironov and A.Morozov, *Phys. Lett. B490* (2000) 173, hep-th/0005280
- [6] A.Morozov and A.Niemi, *Nucl.Phys. B666* (2003) 311-336, hep-th/0304178
- [7] A.Zamolodchikov, *JETP Lett.* 43 (1986) 730; *Sov. J. Nucl. Phys.* 46 (1987) 1090.
- [8] A.Mironov and A.Morozov, *Phys.Lett. B252* (1990) 47-52; *JHEP* 0902 (2009) 024, arXiv:0807.2843; F.David, *Mod.Phys.Lett. A5* (1990) 1019; J.Ambjorn and Yu.Makeenko, *Mod.Phys.Lett. A5* (1990) 1753; H.Itoyama and Y.Matsuo, *Phys.Lett.* 255B (1991) 202
- [9] M.Fukuma, H.Kawai and R.Nakayama, *Int.J.Mod.Phys. A6* (1991) 1385; R.Dijkgraaf, E.Verlinde and H.Verlinde, *Nucl.Phys. B348* (1991) 565
- [10] A.Alexandrov, A.Mironov and A.Morozov, hep-th/0310113; hep-th/0412205; hep-th/0412099
- [11] B.Eynard, *JHEP* 0411 (2004) 031, hep-th/0407261; B.Eynard and N.Orantin, *JHEP* 0612 (2006) 026, math-ph/0504058; math-ph/0702045; L.Chekhov, B.Eynard, *JHEP* 0603 (2006) 014, hep-th/0504116; *JHEP* 0612 (2006) 026, math-ph/0604014; A.Alexandrov, A.Mironov and A.Morozov, hep-th/0605171; hep-th/0608228; arXiv:0906.3305; B.Eynard, M.Mariño and N.Orantin, *JHEP* 0706 (2007) 058, hep-th/0702110; N.Orantin, arXiv:0808.0635; I.Kostov and N.Orantin, arXiv:1006.2028; L.Chekhov, B.Eynard and O.Marchal, arXiv:1009.6007; V.Bouchard and P.Sulkowski, arXiv:1105.2052
- [12] V.Knizhnik and A.Morozov, *JETP Lett.* 39 (1984) 240; H.Levine and S.Libby, *Phys. Lett. B150* (1985) 182

- [13] N.Seiberg and E.Witten, Nucl.Phys. B426 (1994) 19-52 hep-th/9407087; hep-th/9408099
- [14] G.Moore, N.Nekrasov, S.Shatashvili, Nucl.Phys. B534 (1998) 549-611, hep-th/9711108; hep-th/9801061; A.Losev, N.Nekrasov and S.Shatashvili, CMP 209 (2000) 97-121, hep-th/9712241; 77-95, hep-th/9803265
- [15] R. Dijkgraaf and H. Fuji, Fortsch.Phys. 57 (2009) 825-856, arXiv:0903.2084; R.Dijkgraaf, H.Fuji and M.Manabe, arXiv:1010.4542
- [16] A.Mironov, A.Morozov and L.Vinet, Teor.Mat.Fiz. 100 (1994) 119-131, hep-th/9312213; A.Gerasimov, S.Khoroshkin, D.Lebedev, A.Mironov, A.Morozov, IJMPA 10 (1995) 2589, hep-th/9405011
- [17] A.Morozov and L.Vinet, Int.J.Mod.Phys. A13 (1998) 1651-1708, hep-th/9409093
- [18] for a recent review and references see: A.Marshakov, arXiv:1207.1869
- [19] I.Krichever, Doklady Acad. Nauk USSR 227 (1976) 291; Funk. Anal. i Pril. 11 (1977) 15
- [20] A.Morozov, hep-th/9303139; hep-th/9502091; hep-th/0502010; arXiv:1201.4595; arXiv:1204.3953; A.Mironov, hep-th/9312212; hep-th/9409190; hep-th/9409190; q-alg/9711006; A. Mironov, A. Morozov and Z. Zakirova, Phys.Lett. B 711 (2012) 332-335, arXiv:1202.6029
- [21] M.Green, J.Schwarz and E.Witten, *Superstring Theory*, Cambridge University Press, 1987
- [22] V. Zakharov, S. Manakov, S. Novikov and L. Pitaevski, *Soliton theory*, Moscow, Nauka, 1980; B.Dubrovin and S.Novikov, Sov.Math. Doklady 27 (1983) 665 - 669; I.Krichever, Comm.Math.Phys. 143 (1992) 415, hep-th/9205110; B.Dubrovin, Nucl.Phys. B379 (1992) 627; T.Nakatsu and K.Takasaki, Mod.Phys.Lett. 11 (1996) 417, hep-th/9509162; H.Itoyama and A.Morozov, Nucl.Phys. B491 (1997) 529, hep-th/9512161; hep-th/0301136; A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Nucl.Phys. 527B (1998) 690, hep-th/9802007; A.Morozov, hep-th/9903087
- [23] E.Witten, Nucl.Phys. B460 (1996) 335, hep-th/9510135; M.Martellini, A.Marshakov and A.Morozov, Phys.Lett. B418 (1998) 294-302, hep-th/9706050
- [24] D.Gaiotto, arXiv:0904.2715; arXiv:0908.0307
- [25] L.Alday, D.Gaiotto and Y.Tachikawa, Lett.Math.Phys. 91 (2010) 167-197, arXiv:0906.3219; N.Wyllard, JHEP 0911 (2009) 002, arXiv:0907.2189; A.Mironov and A.Morozov, Nucl.Phys. B825 (2009) 1-37, arXiv:0908.2569
- [26] E. Witten, Nucl.Phys. B340 (1990) 281; R.Dijkgraaf, E.Verlinde, and H.Verlinde, Nucl. Phys. B 352 (1991) 59; B.Dubrovin, arXiv:hep-th/9407018; A.Marshakov, A.Mironov and A.Morozov, hep-th/9607109; hep-th/9701014; hep-th/9701123; A.Mironov and A.Morozov, Phys.Lett. B424 (1998) 48, hep-th/9712177; hep-th/9711194; H.Braden, A.Marshakov, A.Mironov and A.Morozov, arXiv:hep-th/0606035
- [27] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov, A.Morozov, Phys.Lett., B355 (1995) 466-477; E.Martinec and N.Warner, Nucl.Phys.B459:97-112,1996, hep-th/9509161; hep-th/9511052; R.Donagi and E.Witten, Nucl.Phys., B460 (1996) 299-334; N.Nekrasov and V.Pestun, arXiv:1211.2240
- [28] D.Galakhov, A.Mironov and A.Morozov, JHEP 2012 (2012) 67, arXiv:1205.4998; M.Billo, M.Frau, L.Gallot, A.Lerda and I.Pesando, arXiv:1302.0686
- [29] M.R.Adams, J.Harnad and J.Hurtubise, Lett.Math.Phys. 20 (1990) 299-308; J.Harnad, Comm.Math.Phys. 166 (1994) 337-365, hep-th/9301076; M.Bertola, B.Eynard and J.Harnad, Comm.Math.Phys. 229 (2002) 73-120, nlin/0108049; L.Bao, E.Pomoni, M.Taki and F.Yagi, arXiv:1112.5228; G.Wilson and J. Reine Angew. Math. 442 (1993) 177-204
- [30] A.Mironov, A.Morozov, B.Runov, E.Zenkevich and A.Zotov, arXiv:1204.0913; arXiv:1206.6349



- [31] A.Marshakov, A.Mironov, A. Morozov, arXiv:0907.3946; arXiv:0909.2052; arXiv:0909.3338  
A.Mironov,S.Mironov,A.Morozov,And.Morozov, Theor.Math.Phys. 165 (2010) 1662-1698, arXiv:0908.2064;  
R.Poghossian, JHEP 0912 (2009) 038, arXiv:0909.3412;  
G. Bonelli and A. Tanzini, arXiv:0909.4031;  
H. Awata and Y. Yamada, arXiv:0910.4431;  
L.Hadasz, Z.Jaskolski and P.Suchanek, arXiv:0911.2353; arXiv:1004.1841;  
V.Alba and And.Morozov, arXiv:0911.0363; Nucl.Phys.B840 (2010) 441-468, arXiv:0912.2535;  
T.Eguchi and K.Maruyoshi, arXiv:0911.4797; arXiv:1006.0828;  
R.Schiappa and N.Wyllard, arXiv:0911.5337;  
V.Alba, V.Fateev, A.Litvinov and G.Tarnopolsky, Lett.Math.Phys.98:33-64,2011, arXiv:1012.1312;  
A. Belavin and V. Belavin, Nucl.Phys.B850:199-213,2011, arXiv:1102.0343;  
A.Belavin, M.Bershtein, B.Feigin, A.Litvinov and G.Tarnopolsky, arXiv:1111.2803;  
O. Gamayun, N. Iorgov and O. Lisovyy, JHEP 10 (2012) 38, arXiv:1207.0787; arXiv:1302.1832
- [32] V.I.Dotsenko and V.Fateev, Nucl.Phys. B240 (1984) 312-348; B 251 (1985) 691;  
S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, S.Pakuliak, N.Phys. B404 (1993) 717, hep-th/9208044;  
A.Mironov and A.Morozov, Phys.Lett. B680 (2009) 188-194, arXiv:0908.2190;  
R.Dijkgraaf and C.Vafa, arXiv:0909.2453;  
H.Itayama, K.Maruyoshi and T.Oota, Prog.Theor.Phys. 123 (2010) 957-987, arXiv:0911.4244;  
A.Mironov, A.Morozov, Sh.Shakirov, arXiv:0911.5721; arXiv:1001.0563; arXiv:1010.1734; arXiv:1012.3137;  
R.Schiappa and N.Wyllard, arXiv:0911.5337;  
T.Eguchi and K.Maruyoshi, arXiv:0911.4797; arXiv:1006.0828;  
A. Mironov, Al. Morozov and And. Morozov, Nucl.Phys.B843:534-557,2011, arXiv:1003.5752;  
A.Morozov and Sh.Shakirov, arXiv:1004.2917;  
K.Maruyoshi and F.Yagi, arXiv:1009.5553;  
A.Mironov, A.Morozov, Sh.Shakirov and A.Smirnov, Nucl.Phys. B855 (2012) 128-151, arXiv:1105.0948
- [33] N.Nekrasov and S.Shatashvili, arXiv:0901.4744; arXiv:0901.4748; arXiv:0908.4052;  
A. Mironov, A. Morozov, JHEP 04 (2010) 040, arXiv:0910.5670; J.Phys.A43:195401,2010, arXiv:0911.2396;  
A.Popolitov, arXiv:1001.1407;  
Wei He, Yan-Gang Miao, Phys.Rev. D82 (2010) 025020, arXiv:1006.1214;  
K.Maruyoshi and M.Taki, arXiv:1006.4505;  
A.Marshakov, A.Mironov and A.Morozov, J.Geom.Phys.61:1203-1222,2011, arXiv:1011.4491;  
N.Nekrasov, A.Rosly, and S.Shatashvili, arXiv:1103.3919;  
F.Fucito, J.F.Morales, R.Poghossian and D. Ricci Pacifici, arXiv:1103.4495;  
Y.Zenkevich, arXiv:1103.4843;  
N.Dorey, T.J.Hollowood and S.Lee, arXiv:1103.5726
- [34] E.Witten, Comm.Math.Phys. 121 (1989) 351-399; arXiv:1001.2933; arXiv:1101.3216;  
D.Gaiotto and E.Witten, arXiv:1106.4789
- [35] E.Guadagnini, M.Martellini and M.Mintchev, In Clausthal 1989, Proceedings, Quantum groups, 307-317;  
Phys.Lett. B235 (1990) 275;  
N.Yu.Reshetikhin and V.G.Turaev, Comm. Math. Phys. 127 (1990) 1-26;  
R.K.Kaul, P.Ramadevi, Commun.Math.Phys. 217 (2001) 295-314, hep-th/0005096;  
Zodinmawia, P.Ramadevi, arXiv:1107.3918
- [36] A.Smirnov, hep-th/09105011;  
A.Morozov and A.Smirnov, Nucl.Phys.B 835:284-313, 2010, arXiv:1001.2003;  
D.Galakhov, A.Mironov, A.Morozov and A.Smirnov, arXiv:1104.2589;  
A. Mironov and A. Morozov, arXiv:1208.2282;  
S.Nawata, P.Ramadevi and Zodinmawia, arXiv:1302.5144
- [37] M.Rosso and V.F.R.Jones, J. Knot Theory Ramifications, 2 (1993) 97-112;  
J.M.F.Labastida and M.Marino, J.Knot Theory Ramifications, 11 (2002) 173;  
X.-S.Lin and H.Zheng, Trans. Amer. Math. Soc. 362 (2010) 1-18 math/0601267;  
S.Stevan, Annales Henri Poincare, 11 (2010) 1201-1224, arXiv: 1003.2861
- [38] R.Lawrence and L.Rozhansky, Comm.Math.Phys. 205 (1999) 287;  
M.Mariño, Comm.Math.Phys. 254 (2004) 25-49, hep-th/0207096;

- C.Beasley and E.Witten, J.Diff.Geom. 70 (2005) 183-323, hep-th/0503126;  
Y.Dolivet and M.Tierz, J.Math.Phys. 48 (2007) 023507, hep-th/0609167;  
A.Brini, B.Eynard and M.Mariño, arXiv:1105.2012
- [39] P.Dunin-Barkowski, A.Mironov, A.Morozov, A.Sleptsov and A.Smirnov, JHEP (2013), arXiv:1106.4305;  
A.Mironov, A.Morozov, Sh.Shakirov and A.Sleptsov, JHEP 2012 (2012) 70, arXiv:1201.3339
- [40] A.Mironov, A.Morozov and And.Morozov, JHEP 03 (2012) 034, arXiv:1112.2654;  
H.Itayama, A.Mironov, A.Morozov, And.Morozov, arXiv:1204.4785; arXiv:1209.6304;  
A.Anokhina, A.Mironov, A.Morozov and And.Morozov, Nucl.Phys. B868 (2013) 271, arXiv:1207.0279
- [41] S.Gukov, A.Schwarz and C.Vafa, Lett.Math.Phys. 74 (2005) 53-74, hep-th/0412243;  
N.Dunfeld, S.Gukov and J.Rasmussen, Experimental Math. 15 (2006) 129-159, math/0505662
- [42] M.Aganagic and Sh.Shakirov, arXiv:1105.5117; arXiv:1202.2489; arXiv:1210.2733;  
I.Cherednik, arXiv:1111.6195;  
Sh.Shakirov, arXiv:1111.7035 and *to appear*;  
A.Negut, arXiv:1209.4242;  
E.Gorsky and A.Negut, *to appear*
- [43] M.Khovanov and L.Rozhansky, Fund. Math. 199 (2008) 1, math.QA/0401268; math.QA/0505056;  
S.Gukov, A.Iqbal, C.Kozcaz and C.Vafa, arXiv:0705.1368;  
S.Gukov, arXiv:0706.2369;  
E.Gorsky, Journal of Singularities, 3 (2011), 48-82, arXiv:0807.0491; arXiv:1003.0916;  
T.Dimofte, S.Gukov and L.Hollands, arXiv:1006.0977;  
S.Gukov and P.Sulkowski, arXiv:1108.0002;  
N.Carqueville and D.Murfet, arXiv:1108.1081;  
S.Gukov and M.Stosic, arXiv:1112.0030;  
A.Oblomkov, J.Rasmussen and V.Shende, arXiv:1201.2115 (with an Appendix by E.Gorsky);  
H.Itayama, A.Mironov, A.Morozov and And.Morozov, JHEP 7 (2012) 131, arXiv:1203.5978;  
H.Fuji, S.Gukov and P.Sulkowski, arXiv:1205.1515; arXiv:1209.1416;  
S.Nawata, P.Ramadevi, Zodinmawia, and X.Sun, arXiv:1209.1409
- [44] A.Mironov, A.Morozov and And.Morozov, arXiv:1112.5754
- [45] A.Mironov, A.Morozov and A.Sleptsov, arXiv:1303.1015
- [46] V.Vassiliev, *Cohomology of knot spaces*, Amer. Math. Soc., Providence, 1990; *Complements of discriminants of smooth maps: topology and applications*, Trans.of Math.Mono. 98, Amer. Math. Soc., Providence, 1992;  
M. Kontsevich, Advances in Soviet Math. 16, part 2 (1993) 137;  
D. Bar-Natan, Topology, 34 (1995) 423-472; q-alg/9607001;  
P.Dunin-Barkowski, A.Sleptsov and A.Smirnov, arXiv:1112.5406; arXiv:1201.0025
- [47] Anton Morozov, JHEP 12 (2012) 116, arXiv:1211.4596; JETP Lett. (2013), arXiv:1208.3544
- [48] A.Anokhina, A.Mironov, A.Morozov and And.Morozov, arXiv:1211.6375
- [49] A.Mironov, A.Morozov and S.Natanzon, arXiv:0904.4227; arXiv:0909.1164; arXiv:1012.0433
- [50] A.Mironov, A.Morozov and S.Natanzon, arXiv:1208.5057; arXiv:1210.6955; arXiv:1212.2041
- [51] M. Khovanov, Duke Math. J. 101 (2000) 359-426;  
D.Bar-Natan, Algebraic and Geometric Topology 2 (2002) 337-370, math/0201043; math/0606318;  
V.Dolotin and A.Morozov, JHEP 1301 (2013) 065, arXiv:1208.4994; arXiv:1209.5109
- [52] R.Dijkgraaf, In: *The moduli spaces of curves*, Progress in Math. 129 (1995), 149-163, Birkhauser
- [53] R.Gopakumar and C.Vafa, Adv. Theor. Math. Phys. 3 (1999) 1415, hep-th/9802016;  
H.Ooguri and C.Vafa, Nucl.Phys. B 577 (2000), 419, hep-th/9912123;  
J.Labastida, M.Marino and C.Vafa, JHEP0011:007, 2000, hep-th/0010102
- [54] K.Liu and P.Peng, arXiv:0704.1526
- [55] S.Kharchev, A.Marshakov, A.Mironov and A.Morozov, hep-th/9203043; hep-th/9208046; hep-th/9312210;  
A.Mironov, A.Morozov and G.Semenoff, Int.J.Mod.Phys. A10 (1995) 2015, hep-th/9404005;  
A.Alexandrov, A.Mironov, A.Morozov and S.Natanzon, arXiv:1103.4100; arXiv:1108.0885